RECONSTRUCTING VERMEER'S PERSPECTIVE IN 'THE ART OF PAINTING'

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ABSTRACT: The reason for reconstructing the perspective in Vermeer's main work *'The Art of Painting'* is to find arguments contra Ph. Steadman's theory that a camera obscura was used for producing a geometrically correct construction. In this paper on the one hand the computer-aided analytic reconstruction is explained, and it is pointed out under which assumptions the reconstruction of all displayed object is possible. On the other hand, the paper focuses on conclusions from this reconstruction. To avoid any misunderstanding, the reason for exposing geometrical flaws in the perspective is not pedantically doctrinaire but shall demonstrate that the laws of composition and artistic intuition stand much higher than just copying and scaling a tiny camera-obscura depiction.
Keywords: Vermeer, perspective, reconstruction, camera obscura.

1. PRELIMINARY STATEMENT

This survey concerning Vermeer's *'The Art of Painting'* does *not* aim to deconstruct the myth of this major work of European art. The intention is to demonstrate – with the help of precise mathematical and geometrical methods – that *the picture-composition is not an imitation of a stage-like scene. The picture suggests a natural reality but its logic underlies that of Vermeer's exactly defined image area.*

2. INTRODUCTION

Johannes Vermeer van Delft painted his most important picture *'The Art of Painting'* in the years 1666/1668. Now it is one of the main attractions in the permanent collection of the Kunsthistorisches Museum Vienna, where a special Vermeer exhibition took place from January to April 2010 [2].

An inspection of this masterpiece reveals that it communicates in painted form a wide spectrum of knowledge referring to the *art of painting:* At the beginning one can see the famous motive of the curtain. Something mysterious is revealed in front of our eyes – although we cannot express absolute truth even after closer inspection and research.

Figure 1: General view of Vermeer's
'De Schilderconst' ['The Art of Painting']

Therefore we take for granted that the central concern of the artist is not the depicted scene but the meaning behind it and the intention to follow certain laws of composition.

It is our ambition to know more about the nature of a great masterpiece and to discover some of Vermeer's tricks and secrets by applying computer-aided methods.

Another reason for reconstructing the perspective of Vermeer's painting is to find arguments contra Philip Steadman's theory [2] that a camera obscura was used for producing a geometrically correct construction (see also [3]). In fact, the perspective of the interior is rather simple.

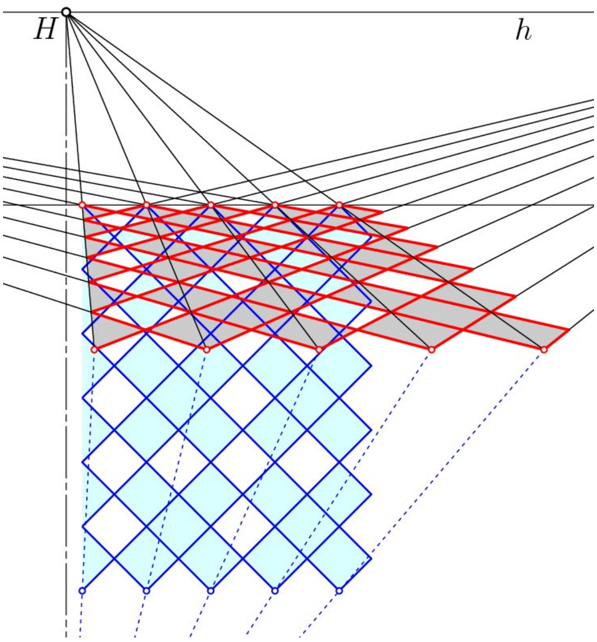

Figure 2: Drawing a perspective of the interior

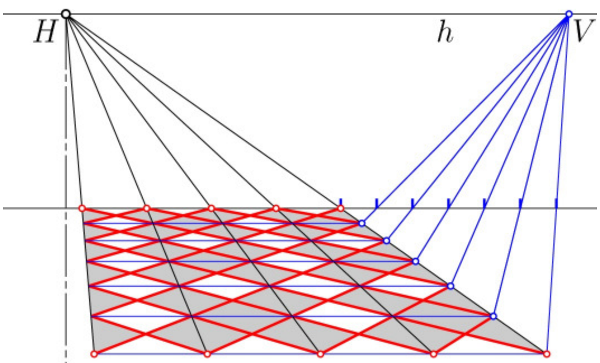

Figure 3: Construction based on two equidistant scales

Fig. 2 shows what is needed to let a quadrangular grid (blue) correspond the perspective image of the tiles (red) in a perspective collineation. In Fig. 3 only two equidistant scales were used to construct the perspective – thus being independent of unattainable vanishing points. Point V is an arbitrary point on the horizon h. Most probably Vermeer used this method since recently (note [6], p. 199) on the original canvas a deformation was detected at the intersection between the horizon h and the right borderline. This point is not the vanishing point of the stool as stated in [6] but Vermeer's choice for V according to Fig. 3.

Finally, starting from the perspective of the tiles the images of the different objects can be constructed in a standard way by protracting the altitudes. Hence, there should be no technical reason for Vermeer to use a camera obscura for obtaining the outlines of the perspective.

Moreover, significant elements of the composition withdraw themselves – by overlapping or veiling – from a precise and uniform concept for the depicted central perspective construction. Therefore not all objects in the scene need to be equally scaled. As an example, the length of the table with the still life reaches around 180 cm; therefore the dimension of the mask would be around 50-60 cm – an object that would not exist that large in reality.

In this research the computer-aided analytic reconstruction was used instead of graphical standard methods. This offers the possibility to vary parameters like the height of the table or the size of the tiles quite easily. Furthermore, in this way also different geometric conditions can be used at the least square fit for detecting the most reasonable dimensions of the depicted objects. The main aim is to discover 'faults' in Vermeer's *suggestion of reality* in contrast to an *imitation of reality* by using a camera obscura.

In Chapter 3 it is explained how the analytic reconstruction was carried out. In Chapter 4 we focus on the depicted objects and list the assumption which were necessary to recover the shape and position of each depicted object. We continue with a summary of arguments against the camera-obscura theory in Chapter 5. Finally, Chapters 6 and 7 reveal that for Vermeer the balance between the depicted objects turns out to be more decisive than to follow the exact rules of perspective.

3. ANALYTIC RECONSTRUCTION

Our reconstruction is based on several assumptions.

Assumption 1: *Vermeer's painting shows a photo-like perspective with an image plane parallel to the back-wall.*

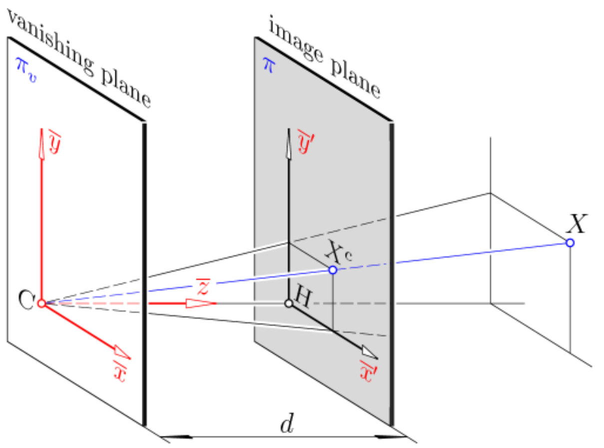

Figure 4: Standard coordinate frames

3.1 Mapping equations

We start using particular coordinate systems (see Fig. 4): The *camera frame* defines the world coordinates $(\overline{x}, \overline{y}, \overline{z})$: This frame has its origin at the projection center C and the $\overline{z}$ -axis as central ray perpendicular to the image plane. The coordinates $(\overline{x}', \overline{y}')$ in the image plane are centered at the central vanishing point H, the intersection point with the central ray. Then the central projection $X \mapsto X^c$ obeys in matrix form the equations (see, e.g., [4])

$$\begin{pmatrix}\overline{x}'\\ \overline{y}'\end{pmatrix} = \frac{d}{\overline{z}}\begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} \text{ with } d = CH\,.$$

Now we adjust the *world coordinates* to the depicted scene (see Fig. 5): The back wall is specified as the yz -plane and it serves also as image plane. The y -axis is horizontal. The x -axis contains diagonals of the most-left black tiles, which are mainly hidden under the table and the front chair; only the most-right vertices of these tiles are visible.

At the beginning we choose half of the diagonal length of the tiles as unit length. Hence the vertices of the tiles have positive integers as world coordinates. The following transformation equations hold between our adjusted world coordinates and the camera frame:

$$\begin{pmatrix}\overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix} = \begin{pmatrix}-y_H\\ -z_H\\ d\end{pmatrix} + \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

Here $(0, y_H, z_H)$ are the world coordinates of H, and (d, y_H, z_H) are that of C.

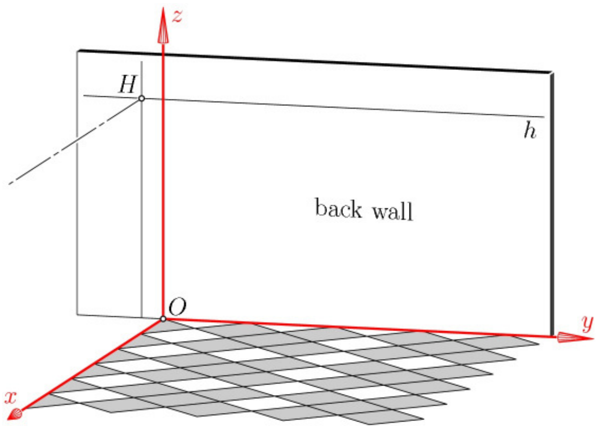

Figure 5: World coordinates in the setting

Since the image plane has been fixed in space, we must admit scaling factors for the image. We use factors σ_x, σ_y between the virtual image in the back-wall and the underlying painting, one in x- and one in y-direction. Furthermore, we translate the original standard coordinates. For our new *image coordinates* (x', y') the origin lies in the left bottom-corner of the painting. When (x'_H, y'_H) denote the image coordinates of the central vanishing point H, then we have the coordinate transformation

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}x'_H\\ y'_H\end{pmatrix} + \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_y\end{pmatrix}\begin{pmatrix}\overline{x}'\\ \overline{y}'\end{pmatrix}.$$

Thus we come up with the mapping equations

$$x' = x'_H + d\sigma_x \frac{-y_H + y}{d - x}$$

$$y' = y'_H + d\sigma_y \frac{-z_H + z}{d - x}$$

of our assumed perspective. There are seven unknowns included, (d, y_H, z_H) as exterior parameters and σ_x, σ_y, (x'_H, y'_H) as interior parameters.

3.2 Reconstruction by a least square fit

Our reconstruction of Vermeer's masterpiece is based on a photograph of size 21.5 × 18.0 cm of the original painting. We scanned this photo and converted it into PostScript. Then we determined the coordinates of image points with the option 'Measure' of GSview. The size of the digital image is 1710.1 × 1441.6 pt. The original painting is of size 120 × 100 cm so that 1 pt in our scanned photo corresponds to about 0.07 cm original size. By the way, the ratio 6 : 5 is preferred by Vermeer; even the depicted canvas poised on the eagle has the same ratio.

There are 18 vertices X_1, ..., X_{18} of tiles visible in the painting. Their (integer) world coordinates $(x_i, y_i, 0)$ (Fig. 5) and their image coordinates (x'_i, y'_i) are available. Hence, each of these grid points gives two equations

$$x'_i u_1 - y_i u_2 + x_i u_3 - u_4 = x_i x'_i,$$
$$y'_i u_1 - z_i u_5 + x_i u_6 - u_7 = x_i y'_i.$$

They are linear in the 7 unknowns $u_1, ..., u_7$, where

$$u_1 = d, \quad u_2 = d\sigma_x, \quad u_3 = \sigma_x x'_H,$$
$$u_4 = d\sigma_x(x'_H - y_H), \quad u_5 = d\sigma_y,$$
$$u_6 = \sigma_y y'_H, \quad u_7 = d\sigma_y(y'_H - z_H).$$

These 36 inhomogeneous equations define an overdetermined system – in matrix form expressable by $\mathbf{A} \cdot \mathbf{u} = \mathbf{b}$. We know that in the sense of least square fit the optimal solution for the unknowns u_1, ..., u_7 solves the system of *normal equations*

$$(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{A}) \cdot \mathbf{u} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{b}.$$

In terms of the Moore-Penrose pseudoinverse $\mathbf{A}^{ps}$ of $\mathbf{A}$ we can express this optimal solution also by $\mathbf{u} = \mathbf{A}^{ps} \cdot \mathbf{b}$. From these optimal u_1, ..., u_7 we compute step by step the external and internal parameters of the given perspective.

3.3 Discussion of the numerical results

The optimal result of our procedure reads as follows: With respect to the original painting, the central vanishing point H has the coordinates (35.4, 55.6) in cm. A deformation in this area in form of a hole, which can be seen as a technical construction aid, was detected 1949 by Hultén [1, p. 199]. Fig. 6 shows point H and the horizon h.

By the way, previous efforts concerning Vermeer's paintings did not deliver any plausible explanation for the placement of the horizon h. Our comment is as follows (compare Fig. 12):

- The horizon, which is relevant for the depiction of the room, passes through the upper part of the painter's body on the level of his heart as well as through his hand which is supported by the maulstick. The brush connects the upper part of the picture with the lower one.
- On the other hand, the depicted painter's horizon, which passes through his eyes, traverses the lower part of the map frame thus connecting the painter with the model. Also Vermeer's signature is placed exactly at this level.
- It is remarkable as well as sophisticated that the horizon of the painter appears higher than that of the beholder of the scene.
- The highest horizon is that of the model Clio. The girl gazes down into the open sketch-book.

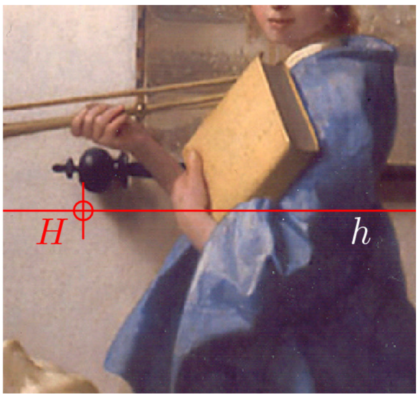

Figure 6: Computed position of the central vanishing point

Of course, the central vanishing point H could also be determined graphically according to Fig. 2 by connecting points of our perspective grid. However, these lines turn out to be by far not collinear; they are rather scattered. A choice of H as a best approximate of the intersection point only does not pay sufficient attention to the strong condition that aligned grid points originate from an equidistant scale as revealed in Fig. 3. This is an additional reason why an analytic method was preferred for the reconstruction.

The distance d between the center C of projection and the image plane (in original size) is 174.9 cm. Therefore the two vanishing points of the sides of the tiles are placed on the horizon h 139.5 cm left from the left edge and 110.4 cm right of the right edge of the painting, respectively. Fig. 3 demonstrates that Vermeer could draw the perspective also without them.

With these numerical results the statement [6, p. 199] cannot be verified with sufficient precision that the *golden ratio* shows up at these vanishing points together with H and the border lines of the painting.

Figure 7: Vertices with maximum deviation, missing corner of a tile (in green) and misplaced seat

When the mapping equations with the optimal parameters are applied to the exact world coordinates of the grid points, we obtain new positions for the 18 vertices. With respect to the original size 120 × 100 cm the *mean errors* in horizontal x- and vertical y-direction are 0.11 cm and 0.08 cm, respectively. The *maximum error* in x-direction is 0.30 cm and that in y-direction 0.24 cm; hence the precision of the depicted tiles is quite remarkable. The grid points with these maximum errors are marked in Fig. 7 by red rings with 1 cm diameter. The computed points are the respective centers of these rings.

4. RECONSTRUCTION OF THE SCENE

After having determined the optimal mapping equations we can proceed by reconstructing the depicted objects as far as this is possible. Already a rough inspection reveals that without additional assumptions many objects cannot be reconstructed because often their relative position to the floor is hidden. This is characteristic for Vermeer and the reason why he sometimes is called *'Sphinx of Delft'*.

The shadows can nowhere be used for recovering information. They are never constructed but serve for contrast effects only.

4.1 The chairs

We recover the placement of the front chair by use of

Assumption 2: *The two depicted chairs, one in front, the other close to the back-wall, are equal models and therefore of the same size.*

It turns out that the reconstructions of both chairs look rather distorted. The corrected edges of the front seat can be seen as dashed lines in Fig. 8.

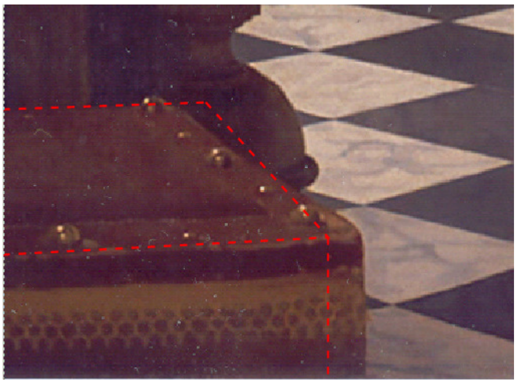

Figure 8: Recovering the chairs

4.2 The stool

The points where the legs of the stool meet the tiled floor form a rather precise rectangle. However, the reconstruction of the top gives a rather distorted rectangle (Fig. 9). An inspection of Vermeer's painting reveals that the image of the stool is closer to an axonometric view than to a perspective because the front edge of the top rectangle is almost parallel to the line connecting the bases of the two front legs as well as to the crossbar between (note Fig. 7). This might either be caused by the fact that the corresponding vanishing point is about 5.4 m left of the left border line of the painting. Or it was Vermeer's intention to mix central and parallel projection in his painting. Or – as pointed out in Chapter 6 – the laws of the plane had priority.

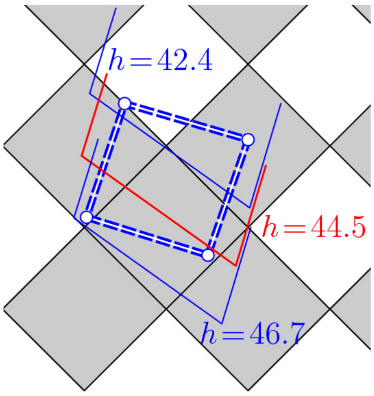

Figure 9: Recovering the seating area of the stool

We can recover the height of the seat by
Assumption 3: *The seat of the stool is positioned symmetrically over the legs.*
As shown in Fig. 9, when varying the height of the stool the position of the seating area varies. The different heights listed in Fig. 9 correspond to a tile length of 27.5 cm (see also Table 1). Assumption 3 leads to a good estimate of the height.

4.3 **The table with the still life**

The points where the legs of table meet the floor are not visible. Since also the exact height of the table is unknown, we cannot figure out the exact position. We only know that there must be sufficient space between the table and the back-wall for Clio and in front between the chair and the table for the curtain hanging down.
Another part of the mysterious masterpiece shall be mentioned here: The opened sketch-book, which is partly protruding beyond the table, seems to touch the artist. When compared with the original painting, it is not certain that the harem pants are overlapping the parchment edge or vice versa – though it is evident in the top view (Figs. 10 and 11) that the table and the parchment are clearly situated in front of the sitting artist. A spot of light set amidst hinders a conclusion to be drawn from the painting itself. Hence, the study in the sketch-book (inspired by the muse Clio?) and the executing artist are directly 'spot-welded' on the image area.

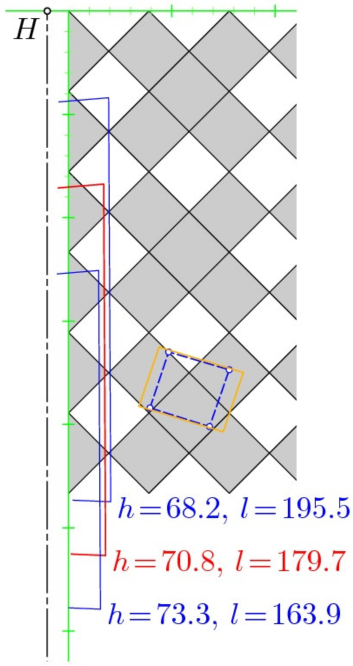

Figure 10: Recovering position and height of the table

4.4 The most probable size of tiles

When comparing the size of the chairs and the stool, the seat of the stool is larger than that of the chairs. This indicates already that the scale of the stool with the painter is slightly greater than that of other depicted of objects.
Despite of all our assumptions, we are not able to figure out the *true size* of the depicted objects. But we express it relatively to the length

of the tiles. Ph. Steadman has good reason for the estimate 29.5 cm (note also the tables in [5, pp. 171-176] or [2, p. 164]). But also an estimate of 27.5 cm gives reasonable results. The following Table 1 lists some recovered dimensions for both choices. Furthermore, it offers a comparision with some confirmed original data listed in the last column.

Table 1: Recovered dimensions in dependance from the length of the tiles (in cm)

assumed length of tiles	27.5	29.5	*original size*
length of table	179.7	192.7	189-192
height of table	70.8	75.9	78-80
thickness of plate	9.4	10.1	8-10
height of stool	44.5	47.8	
width of stool	42.6	45.7	
height of chairs	48.8	52.4	47
width of chairs	32.3-33.9	34.7-36.4	
length of chairs	32.3-33.9	34.7-36.4	
height of chandelier	64.9	69.6	65
diameter of chandelier	75.7	81.2	73
height of Clio	145.0	155.0	
height of sitting painter	130.0	139.5	
size of proper map	95.5 × 133.3	102.4 × 143.0	111.6 × 150.3
total size of wall map	123.5 × 187.7	132.5 × 201.4	147.0 × 211.6

5. ARGUMENTS AGAINST THE CAMERA-OBSCURA THEORY

Here we summarize arguments which in the authors' opinion contradict the statement that Vermeer used a camera obscura for constructing the perspective drawing in *'The Art of Painting'*.

- Primary is the argument that for Vermeer the sense behind the depicted scene, his allegoric allusions and the laws of composition range much higher than the demand for a geometrically exact depiction. The following chapters will demonstrate how laws of composing the painting area dictated the placement of several objects. Note, e.g., the missing part of a black tile right of the painter's right calf (see green lines Fig. 7): The effect of a small black area here would be disturbing.
- It can be questioned that with the technology of the 17th century a manually or mechanically scaled camera obscura projection can reach the remarkable precision with mean error of about ±1 mm at the grid of tiles in Vermeer's painting. Since in this projection the central vanishing point would be in the center, the scaled projection needs the size 130 × 130 cm in order to include the decentral painting of size 120 × 100 cm. Under this assumption, what would be the meaning of the hole at the central vanishing point H ?

If Vermeer had based his painting only on a camera-obscura projection, he hadn't made the errors in the perspective of the stool (Fig. 7) and the front chair (Fig. 8). In particular, the stool lies rather central; so this error cannot be explained by a distortion caused by the lens.

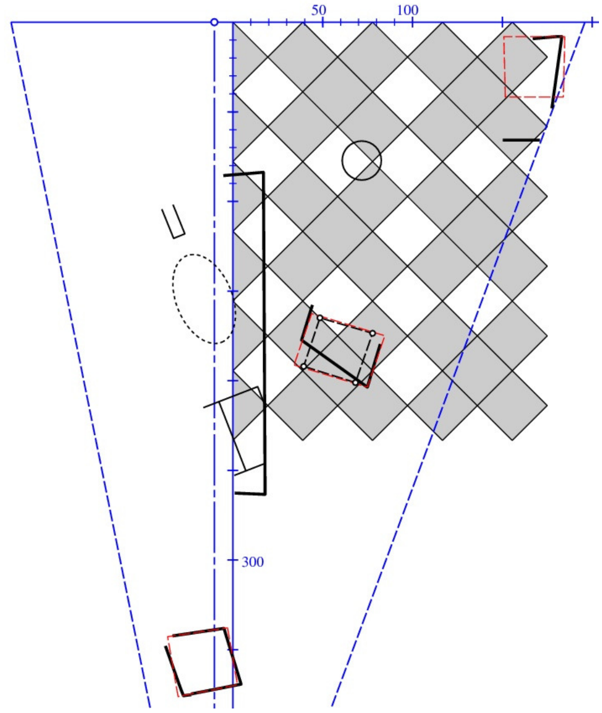

Figure 11: Top view of the depicted scene (with the corrected chairs and stool in red)

6. LAWS OF THE PLANE

Some examples in Vermeer's painting demonstrate that the depicted objects were positioned layer above layer in order to unclear their real dimensions and to veil their stereometrical position with respect to the depicted room. In this way he gets some freedom to place lines according to the *‘laws of the plane’*.

6.1 Harmonical (rational) divisions

Vermeer often used the format with the ratio 12:10 for his paintings. We reveal a more logical correspondence when we uniformly subdivide the side lengths into 12 and 10 units, resp., and place a quadratic grid over the composition (Fig. 12).

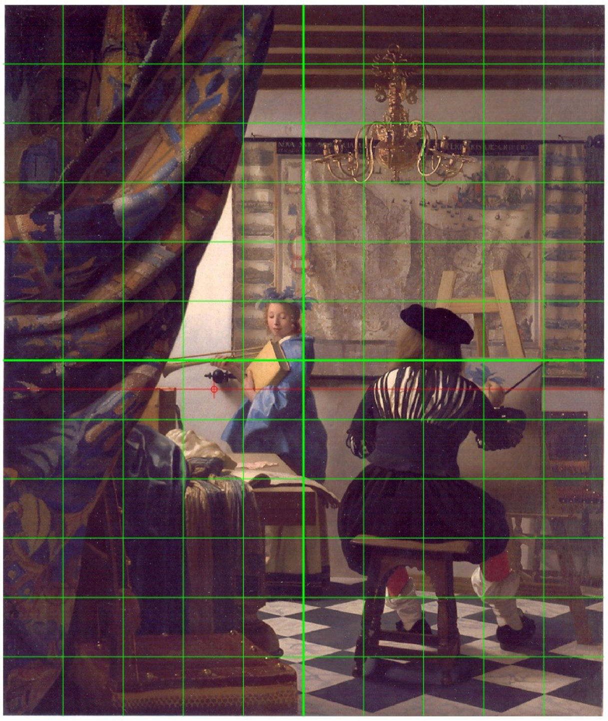

Figure 12: A quadratic grid subdividing the painting area 12:10

- The horizontal center line touches the knob of the red cushioned painting-stick. And it passes through the upper edge of the painting on the easel as well as through the trumpet-holding hand of the girl.
- The vertical center line cuts through the roman number XVII which can be seen in the headline of the map. This might remind of the separation of the Netherlands 1581, when the 17 provinces where subdivided into the 7 Protestant northern provinces and the 10 catholic provinces of Spanish Netherlands.
- This vertical center line covers also the border between the light and dark upper part of the girl's blue cape, the vertical wrinkles of her skirt and the most-left visible vertex of the front-tiles.
- The last partitioning vertical line on the right hand side is a border line for the views of the vedutas on the wall-map. Furthermore, it coincides with the right border of the canvas on the easel and passes through the most-right visible vertex of the front-tiles.

Lines in a painting which produce major connections between several depicted objects are called *‘transparent lines’*. They are of fundamental importance for the formal coherence of the composition.

6.2 The golden ratio

Figure 13: Subdividing the painting area in the golden ratio by horizontal and vertical lines

- When the width of the painting is subdivided in the golden ratio (Fig. 13), the left partitioning line passes exactly through the left border of the wall-map.
- Subdivision of the height defines a line

- which passes approximately through the upper end of the easel. The depicted painter's right ellbow rests on the lower partitioning line.
- In the depicted scene the artist seems to paint on his canvas exactly the *'central motive'*, i.e., the part which is enclosed by these golden partitioning lines.

6.3 The pentagon construction

We incribe a regular pentagon in the circumcircle of the painting. When the highest vertex of the pentagon is chosen on the vertical center line of the painting, we notice (Fig. 14):

- The left hand diagonal passing through the top vertex indicates the inclination of opened curtain.
- The second diagonal passing through the left bottom vertex of the pentagon coincides with the maulstock.
- The line connecting the right bottom vertex with the central vanishing point covers one edge of the table.
- The city of Delft on the map coincides with an intersection point of two diagonals. By the way, this point subdivides the horizontal diagonal segment in the golden ratio.

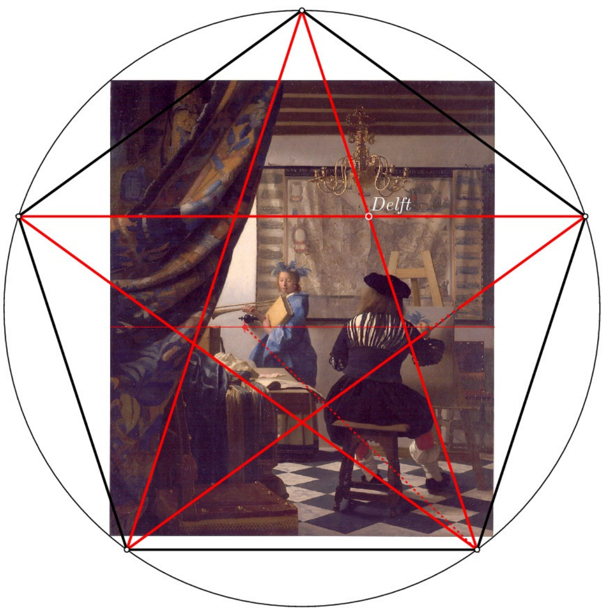

Figure 14: The painting area and its relation to a regular pentagon

7. PRIORITIES OF THE PAINTINGS' COMPOSITION

Whenever consequences of the central perspective construction come in conflict with the plane composition, *great masters prefer the latter*. In this sense, also the relation of the depicted objects to the border lines of the picture has priority over the laws of perspective.

- The front tiles of the floor clearly end approximately 3 mm above the picture border. Vermeer refrains from continuing the design towards the front.
- The shadows in the picture seem to be randomized. Vermeer used them for his compositional needs. E.g.: the composition of the right hand side of the picture is terminated by the dark shadow placed on the right hand side of the map as well as by the shadow in front of the chair next to the wall.
- The wood beams of the ceiling are constructed demonstratively plain. They seem to be folded inside the image plane and define the top of the picture.
- The missing corner of a black tile (Fig. 7) between the painter's right shinbone and the cross bar of the easel shows that the distribution or light and dark had priority. Otherwise, this small black triangle would be disturbing.
- The contour lines of objects in Vermeer's paintings are uniformly blurred *'sfumato'*-like. As a consequence, spatial distances are hard to estimate, the compositions look planar.

CONCLUSION

It was our aim to disclose some of the secrets hidden in Vermeer's masterpiece. For this purpose we applied geometric and computer-aided methods of reconstruction. However, without a few assumptions it is not possible to recover the whole scene. Nevertheless, on the one hand the precision of the depicted tiles is remarkable. On the other hand, we notice different scales for different objects (compare in Table 1 the reconstructed dimensions with some confirmed original sizes).

At several places one can observe that for Vermeer the laws of composing the area in a painting are of higher importance than a geometrically exact construction. The recovered 'flaws' in Vermeer's painting are not at all caused by missing knowledge of geometric rules, but they can only be understood as consequences of Vermeer's method of composition. Hence, they are correct – even in the geometric sense.

Our observations helped also to obtain a clear answer to the question whether a camera obscura was used for the composition in 'The Art of Painting': For Vermeer it was not possible to copy something from a model (note top view in Fig. 11) which does not exist in reality.

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